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Superluminal electromagnetic focus wave modes

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Abstract. A particular transformation of coordinates, associated with superluminal X-pulses, leaves the wave equation invariant and changes focus wave modes into superluminal focus wave pulses. Rather simple and manageable expressions for TM electromagnetic waves allow the investigation of these new localized solutions of Maxwell's equations.

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1 Introduction

Much work over the past fifteen years has been devoted to localized wave solutions of the scalar wave equation [1–7]. Some of these pulses carry a finite energy and since they describe ultrawide band, spatially localized, slowly decaying transmission of energy, they have found many applications in various areas ranging from remote sensing and communications to solid state physics and medical imaging provided of course that one is able to build physical devices to launch these pulses. In free space, the only situation considered here,there exist three classes of localized solutions: focus wave modes (FWM) [8,9], superluminal X-pulses [10,11], subluminal X-pulses [12,13]. We show here that scalar and Maxwell's equation allow superluminal focus wave modes as solutions.

Electromagnetic FWM's have been previously obtained either from the Hertz vector potential [3] or from the complex representation $\mathbf{E}+\mathrm{i}\mathbf{H}$ of the electromagnetic field [14,16]. For electromagnetic focus FWM's, we use an approach different to [3,16]. Let $A_x(\mathbf{x},t)$, $A_y(\mathbf{x},t)\,\{\mathbf{x}=(x,y,z)\}$ be two arbitrary solutions of the scalar wave equation DA $=0$. The components of the solutions to Maxwell's equations are

$$\begin{aligned}
&E_x=-1/c\,\partial_t\partial_z A_x,\\
&E_y=-1/c\,\partial_t\partial_z A_y,\\
&E_z=1/c\,\partial_t(\partial_x A_x+\partial_y A_y)\\
&H_x=\partial_x\partial_y A_x+(\partial_y^2+\partial_z^2)A_y,\\
&H_y=-\partial_x\partial_y A_y-(\partial_x^2+\partial_z^2)A_x,\\
&H_z=\partial_z(\partial_y A_x-\partial_x A_y).
\end{aligned}\tag{1}$$

We assume:

$$A_y=0,\quad 1/c\,\partial_z A_x=B_x,\quad \partial_y B_x=0\tag{2}$$

in which B_x is a localized wave solution of the scalar wave equation supplying localized electromagnetic pulses. Since B_x does not depend on y, we obtain from (2) the TM waves

$$\begin{aligned}
&E_y=H_x=H_z=0,\quad E_x=-\partial_z B_x(\mathbf{u},t),\\
&E_z=\partial_x B_x(\mathbf{u},t),\qquad H_y=-1/c\partial_t B_x(\mathbf{u},t)
\end{aligned}\tag{3}$$

where $\mathbf{u}=(x,z)$. From now on, we concentrate on the TM waves (3) which is sufficient to describe the main features of the electromagnetic FWM's and we are mainly concerned with the research of finite energy weighted TM and superluminal pulses.

2 Electromagnetic focus wave modes

2.1 TM-pulses

The 2D-scalar FWM's have the form [16] in which a,b are positive constants

$$\psi(\mathbf{u},t)=w^{-1/2}\exp\left[\mathrm{i}b(z+ct)-bx^2/w)\right],\qquad w=a+\mathrm{i}(z-ct).\tag{4}$$

Substituting (4) into (3) gives with $B_x=\mathrm{i}\psi$ the TM focus wave modes

$$\begin{aligned}
&E_x(\mathbf{u},t)=w^{-1/2}e_x(b)\exp(-b\Omega),\quad e_x(b)=b(1+x^2/w^2)-1/2w\\
&E_z(\mathbf{u},t)=w^{-1/2}e_z(b)\exp(-b\Omega),\quad e_z(b)=-2\mathrm{i}bx/w\\
&H_y(\mathbf{u},t)=w^{-1/2}h_y(b)\exp(-b\Omega),\quad h_y(b)=-b\left(1-x^2/w^2\right)-1/2w
\end{aligned}\tag{5}$$

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in which

$$\Omega = x^2/w - \mathrm{i}(z+ct). \tag{6}$$

In order to check the energy carried on by FWM's we are interested in the TM weighted FWM's known as splash wave modes [3,16,18]

$$\begin{aligned} E_{F,x}(\mathbf{u},t) &= \int_0^\infty \mathrm{d}bF(b)\, w^{-1/2} e_x(b) \exp(-b\Omega) \\ E_{F,z}(\mathbf{u},t) &= \int_0^\infty \mathrm{d}bF(b)\, w^{-1/2} e_z(b) \exp(-b\Omega) \\ H_{F,y}(\mathbf{u},t) &= \int_0^b \mathrm{d}bF(b)\, w^{-1/2} h_y(b) \exp(-b\Omega) \end{aligned} \tag{7}$$

where $F(b)$ is a weight function which should supply pulses with finite energy. For instance, with $F(b) = J_0(bk)$ in which J_0 is the Bessel function of the first kind of order zero and k a positive scalar, the integrals (7) become Lipschitz-Hankel integrals [17] with an explicit expression given in [18], but as discussed in Section 3 and Appendix A, their energy is not finite.

2.2 Superluminal TM-pulses

Let us introduce the coordinates

$$\zeta = g(vz/c - ct), \quad c\tau = g(z - vt), \quad g^{-2} = v^2/c^2 - 1 \tag{8}$$

then, substituting ζ, τ, to z, t, into (4) gives the superluminal focus wave modes

$$\phi(\mathbf{u},t) = \omega^{-1/2} \exp\left[\mathrm{i}bj(z-ct) - bx^2/\omega)\right] \tag{9}$$

in which with $jm = 1$

$$j = g(v/c+1), \quad m = g(v/c-1), \quad \omega = a + im(z+ct). \tag{10}$$

Substituting $\phi(\mathbf{u},t)$ into (3) gives the TM superluminal FWM's

$$\begin{aligned} E_x(\mathbf{u},t) &= \omega^{-1/2}\varepsilon_x(b)\exp(-b\Pi), \\ &\qquad \varepsilon_x(b) = b(j + mx^2/\omega^2) - m/2\omega \\ E_z(\mathbf{u},t) &= \omega^{-1/2}\varepsilon_z(b)\exp(-b\Pi), \quad \varepsilon_z(b) = -2\mathrm{i}bx/\omega \\ H_y(\mathbf{u},t) &= \omega^{-1/2}\xi_y(b)\exp(-b\Pi), \\ &\qquad \xi_y(b) = -b(j - mx^2/\omega^2) - 1/2\omega \end{aligned} \tag{11}$$

with

$$\Pi = x^2/\omega^2 - ij(z-ct). \tag{11a}$$

Superluminal splash wave modes are obtained as in (7) through a weight function $F(b)$.
Remark: using (8) and the substitutions $a = 0$, $g = -\mathrm{i}\gamma$, $b = -\mathrm{i}\beta$ implying $\sigma = \gamma(z - vt)$ in (4) gives the subluminal FWM's

$$\phi^<(\mathbf{u},t) = \omega_0^{-1/2} \exp\left[-\mathrm{i}\beta\gamma(1+v/c)(z-ct) - \mathrm{i}\beta x^2/\omega_0\right] \tag{12}$$

with

$$\omega_0 = a - \gamma(1 - v/c)(z+ct) \tag{13}$$

but the singularities of these expressions make subluminal pulses uninteresting.

3 Finite energy FWM's

Most of FWM's carry an infinite energy which is not a drawback *per se*: plane wave solutions also share this property. Nevertheless to satisfy the practical objectives mentioned in the introduction, one should be able to generate finite energy pulses. Then, we investigate the electromagnetic energy carried by FWM's.

$$U = \iint_0^\infty \mathrm{d}x\mathrm{d}z \left(|E_{F,x}|^2 + |E_{F,z}|^2 + |H_{F,y}|^2\right). \tag{14}$$

3.1 Splash wave modes (7)

We get in Appendix A

$$U = 4\pi^{3/2}(2a)^{-1/2} \int_0^\infty \mathrm{d}b(b^{3/2} + b^{1/2}/4a)|F(b)|^2. \tag{15}$$

So, U is finite if the following integrals are bounded for $n = 1, 3$

$$T_n = \int_0^\infty \mathrm{d}b\, b^{n/2}|F(b)]^2. \tag{15a}$$

As a simple example, we consider the Bessel weight functions $F_{\mu\nu}(bk) = J_\nu(bk)b^{\mu-1}$ where k is a positive scalar and, the integrals (15a) therefore become discontinuous Weber-Schafhteilin integrals [17]

$$T_{n,\mu\nu} = \int_0^\infty \mathrm{d}b[J_\nu(bk)]^2/b^{2-2\mu-n/2} \tag{16}$$

which converge for $2\nu + 1 > 2 - 2\mu - n/2 > 0$ that is for $\mu + \nu > 1/4$ and $\mu < 1/4$ for $n = 1, 3$. Writing $\alpha = 2 - 2\mu - n/2$ the exponent of b, one has [17] when the previous conditions are fulfilled

$$\begin{aligned} T_{n,\mu\nu} = {} & (k/2)^{\alpha-1}\Gamma(\alpha)\Gamma(\nu - \alpha/2 - 1/2) \\ & \times \left[2\Gamma^2\{(\alpha+1)/2\}\Gamma(\nu + \alpha/2 + 1/2\right]^{-1} \end{aligned} \tag{17}$$

in which Γ is the gamma function. The splash wave modes (7) with $F(b) = J_0(bk)$ carry an infinite energy since the conditions for the convergence of (16) are not satisfied. But the integrals (15a) are bounded for exponential weight functions so that there exists a great diversity of finite energy splash wave solutions of Maxwell's equations.

3.2 Superluminal splash wave modes (11)

We prove in Appendix B that weighting (11) as in (7), the electromagnetic energy is $U = U_1 + U_2$ with

$$U_1 = 4\pi^{3/2}(2a)^{-1/2}\int_0^\infty \mathrm{d}b[(2-m^2)b^{3/2} + m^2 b^{1/2}/4a + p(b)]|F(b)|^2 \quad (18)$$

$$p(b) = (2a)^{1/2}b^2(j^2+m^2-2)\int_0^\infty \mathrm{d}y\exp(-y)(y+2ab)^{-1/2} \quad (18a)$$

and with $h = aj/m - 2ijct$, $Reh > 0$

$$U_2 = 2\pi^{3/2}[m^2-1)/am]\iint_0^\infty bb'\,\mathrm{d}b\,\mathrm{d}b' \times F(b)F^*(b')\exp(-h|b-b'|). \quad (19)$$

But:

$$|p(b)| \le (2a\pi)^{1/2}b^2(j^2+m^2-2)\,\exp(-2ab). \quad (20)$$

So, U_1 is finite if the integrals (15a) for $n = 1,3$ and the integral $\int_0^\infty \mathrm{d}b\,b^2\exp(-2ab)\,|F(b)|^2$ are bounded. Now, for $b > b'$, we have $\exp(hb') < \exp(hb)$ so that

$$U_2 \le 2\pi^{3/2}[m^2-1)/am]\int_0^\infty b\,\mathrm{d}bF(b)\int_0^\infty b'\,\mathrm{d}b'F^*(b') \quad (21)$$

and using the approximation $\int_0^\infty b'\mathrm{d}b'F^*(b') = b^2/2F^*(b)$ gives

$$U_2 \le 2\pi^{3/2}[m^2-1)/am]\int_0^\infty b^3/2\mathrm{d}bF(b). \quad (21a)$$

So U is finite if (15a) is bounded for $n = 6$.

Thus the conditions for finite energy TM superluminal FWM's are more stringent than those for the TM pulses (6) but they are also satisfied by exponential weight functions.

4 Conclusions

Using the simple TM waves (3) rather than the general components (1) avoids cumbersome calculations (leaving aside complex media requiring more developments [22–25]) with no consequence on the physical understanding of electromagnetic localized wave propagation.

An important question is whether it is possible to build a physical device able to launch finite energy electromagnetic and scalar localized waves. First, according to relativity, no information can propagate with superluminal velocities. So, the answer to this question is clearly no for superluminal FWM,s as for superluminal X-waves [26] which is not in contradiction with the superluminal behaviors in wave propagation observed in some experiments [27,28] since in fact only the peaks of the waves travel (for a while) with superluminal group velocities [26,29] while the front travels at the velocity of light (see in particular [30,31] where superluminal processes are thoroughly analysed and explained). Note that it was proved some years ago [32] that one may generate (approximate) acoustical FWM's with better performances than usual Gaussian beams. There is no doubt that to launch electromagnetic FWM's is more challenging!

Incidently, according to a recent theory [33], the cosmic ray bursts recently observed by astrophysicists would come from the desexcitation of energetic electrons generated by the collapse of massive black holes and it has been shown [34] that such a desexcitation process can produce focus wave modes. Could gamma ray bursts be made of FWM's ?

The results obtained in this paper for 2D-pulses can be generalized to 3D-pulses. The scalar FWM's [8,9] and X-pulses [5] have the form with $\mathbf{x} = (x,y,z)$ and $r^2 = x^2+y^2$

$$\psi(\mathbf{x},t) = w^{-1}\exp[ib(z+ct) - br^2/w)], \quad w = a + \mathrm{i}(z-ct) \quad (22)$$

and with the variables of (8)

$$\Phi(\mathbf{x},t) = (r^2+\sigma^2)^{-1/2}\exp[ib\zeta - b(r^2+\sigma^2)], \quad \sigma = a + ic\tau \quad (23)$$

while the scalar superluminal focus wave modes utilise the notations of (10)

$$\phi(\mathbf{x},t) = \omega^{-1}\exp\left[ibj(z-ct) - br^2/\omega)\right]. \quad (24)$$

But to get the electromagnetic FWM's and superluminal FWM's, one has now to work with the full equations (1) making calculations a bit more intricate.

Using (8), a referee has obtained the two families of solutions ($n = 1,2$) in which F is an arbitrary differentiable function

$$\Psi(\mathbf{x}) = A_n^{-1}F(b\Theta_n) \quad (25)$$

$$A_1 = (j/m)^{1/2}(z+ct) + a_0,$$
$$A_2 = \left[(z-\beta ct)^2 + (1-\beta^2)r^2\right]^{1/2}, \quad \beta = v/c \quad (25a)$$
$$\Theta_1 = (j/m)^{1/2}(z-ct) + r^2/A_1, \quad \Theta_2 = ct - z \pm A_2 \quad (25b)$$

in which a_0 is a constant with j,m, given by (10) and leading possibly, with appropriate F function, to localized solutions with a better focusing degree than the Brittingham-Kiselev FWM's.

We postpone to a later work the discussion of superluminal 3D-FWM,s with finite energy.

Appendix A

The electromagnetic energy U of TM pulses is

$$U = \iint_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}z I(x,z),$$
$$I(x,z) = \left(|E_{F,x}|^2 + |E_{F,z}|^2 + |H_{F,y}|^2\right) \quad \text{(A.1)}$$

with, according to (7), the asterisk denoting the complex conjugation

$$I(x,z) = |w|^{-1} \iint_0^{\infty} \mathrm{d}b\, \mathrm{d}b'\, F(b)F^*(b') \times \exp(-b\Omega - b'\Omega^*)P(b,b') \quad \text{(A.2)}$$

in which $w = a + \mathrm{i}(z-ct)$, $\Omega = x^2/w - \mathrm{i}(z+ct)$ and

$$P(b,b') = e_x(b)e_x^*(b') + e_z(b)e_z^*(b') + h_h(b)h_y^*(b'). \quad \text{(A.3)}$$

A simple calculation gives

$$\exp(-b\Omega - b'\Omega^*) = \exp[\mathrm{i}(b-b')(z+ct)]\exp(-\lambda x^2/|w|^2) \quad \text{(A.4)}$$

where

$$\lambda = a(b+b') + \mathrm{i}(b-b')(z-ct), \quad |w|^2 = a^2 + (z-ct)^2. \quad \text{(A.5)}$$

Substituting (A.4) into (A.2) gives

$$I(x,z) = \iint_0^{\infty} \mathrm{d}b\, \mathrm{d}b' F(b)F^*(b')\exp[\mathrm{i}(b-b')(z+ct)]G(x) \quad \text{(A.6)}$$

$$G(x) = |w|^{-1}P(b,b')\, \exp\left(-\lambda x^2/|w|^2\right). \quad \text{(A.7)}$$

Taking into account (A.6) and exchanging the order of integrations we get from (A.1)

$$U = \iint_0^{\infty} \mathrm{d}b\, \mathrm{d}b' F(b)F^*(b') \int_{-\infty}^{\infty} \mathrm{d}z \exp[\mathrm{i}(b-b')(z+ct)] \times \int_{-\infty}^{\infty} \mathrm{d}x G(x). \quad \text{(A.8)}$$

Now substituting the expressions (5) of e_x, e_z, h_y, into (A.3) gives

$$\begin{aligned} P(b,b') &= 2bb'(1 + x^4/|w|^4) + 1/2|w|^2 \\ &\quad - (b/w + b'/w^*)x^2/|w|^2 + 4bb'x^2/|w|^2 \\ &= 2bb' + 1/2|w|^2 + (4bb' - \lambda|w|^{-2})x^2/|w|^2 \\ &\quad + 2bb'x^4/|w|^4 \end{aligned} \quad \text{(A.9)}$$

where we used the relation $b/w + b'/w^* = \lambda|w|^{-2}$. So, according to (A.7) and (A.9)

$$\int_{-\infty}^{\infty} \mathrm{d}x G(x) = \int_{-\infty}^{\infty} \mathrm{d}x |w|^{-1} \left[2bb' + 1/2|w|^2 + \left(4bb' - \lambda|w|^{-2}\right)x^2/|w|^2 + 2bb'x^4/|w|^4\right] \exp\left(-\lambda x^2/|w|^2\right) \quad \text{(A.10)}$$

and using the integrals [20]

$$\int_{-\infty}^{\infty} \mathrm{d}x\, x^{2n} \exp(-\lambda x^2/|w|^2) = 1.3...(2n-1)(\pi/\lambda)^{1/2}|w|(|w|^2/2\lambda)^n \quad \text{(A.11)}$$

we get

$$\int_{-\infty}^{\infty} \mathrm{d}x G(x) = 2bb'(\pi/\lambda)^{1/2}\left(1 + \lambda^{-1} + 3\lambda^{-2}/4\right). \quad \text{(A.12)}$$

Taking into account (A.12) and introducing the variable $\theta = z - ct$ the expression (A.8) becomes

$$U = 2\pi^{1/2} \iint_0^{\infty} \mathrm{d}b\, \mathrm{d}b'\, bb'F(b)F^*(b')J(b,b') \quad \text{(A.13)}$$

$$J(b,b') = \exp[2\mathrm{i}(b-b')ct] \int_{-\infty}^{\infty} \mathrm{d}\theta \exp[\mathrm{i}(b-b')\theta]R(\theta) \quad \text{(A.14)}$$

with

$$R(\theta) = \lambda^{-1/2}1 + \lambda^{-1} + 3\lambda^{-2}/4, \quad \lambda \equiv \lambda(\theta) = a(b+b') + \mathrm{i}(b-b')\theta. \quad \text{(A.15)}$$

We follow from now on Ziolkowski in the Appendix of reference [3]. The θ-integration in (A.14) can be performed as a contour integral and the singularity of the integrand is at the point θ_s with $\theta_s = \mathrm{i}[(b'+b)(b'-b)^{-1}]\,a$. Suppose first $b' \neq b$, the sign of $(b'-b)$ determines whether θ_s is located in the upper or lower half of the complex θ-plane. It is in the upper half plane if $b'-b > 0$ and the lower half plane if $b'-b < 0$. However, to ensure the proper behavior at of the integrand at the infinity so that the contour can be closed in the upper half-plane, one must have $b'-b < 0$ and $b'-b > 0$. Consequently because θ_s is not contained within the closed contour the θ-integral is zero for $b \neq b'$ and (A.13) becomes

$$U = 2\pi^{1/2} \int_0^{\infty} \mathrm{d}b\, b^2|F(b)|^2[J(b,b')]_{b=b'} \quad \text{(A.16)}$$

where with $b' - b = \varepsilon$

$$[J(b,b')]_{b=b'} = \lim_{\varepsilon \Rightarrow 0} \int_{-\infty}^{\infty} \mathrm{d}\theta \exp[\mathrm{i}\varepsilon\theta]R(\varepsilon\theta) \quad \text{(A.17)}$$

in which $R(\varepsilon\theta)$ is obtained from (A.15) with $b'-b$ changed into ε in λ. And still using the Ziolkowski's generalization [3] of a previous result [18] we get according to (A.15)

$$[J(b,b')]_{b=b'} = 2\pi \int_0^{\infty} \mathrm{d}y \exp(-y)\Big[(y+2ab)^{-1/2} + (y+2ab)^{-3/2} + 3(y+2ab)^{-5/2}/4\Big]. \quad \text{(A.18)}$$

Integrating by parts gives

$$\int_0^\infty \mathrm{d}y \exp(-y)[(y+2ab)^{-3/2} = 2(2ab)^{-1/2} - 2\int_0^\infty \mathrm{d}y \exp(-y)[(y+2ab)^{-1/2}$$

$$\int_0^\infty \mathrm{d}y \exp(-y)[(y+2ab)^{-5/2} = 2/3(2ab)^{-3/2} - 4/3(2ab)^{-1/2} + 4/3\int_0^\infty \mathrm{d}y \exp(-y)[(y+2ab)^{-1/2}. \tag{A.19}$$

Substituting (A.19) into (A.18) gives

$$[J(b,b')]_{b=b'} = 2\pi\left[(2ab)^{-1/2} + 1/2(2ab)^{-3/2}\right. \tag{A.20}$$

and taking into account (A.20) we get finally from (A.16)

$$U = 4\pi^{3/2}(2a)^{-1/2}\int_0^\infty \mathrm{d}b\, b^2\,|F(b)|^2\left(b^{3/2} + b^{1/2}/4a\right). \tag{A.21}$$

Appendix B

The electromagnetic energy U of the superluminal splash waves obtained by weighting (11) as in (7) is, according to (10) and (11a)

$$U = \iint_{-\infty}^\infty \mathrm{d}x\mathrm{d}z|\omega|^{-1}\iint_{-\infty}^\infty \mathrm{d}b\,\mathrm{d}b' F(b)F^*(b') \times \exp(-b\Pi - b'\Pi^*)P(b,b') \tag{B.1}$$

where $P(b,b')$ is (A.3) with e_x, e_z, h_y transformed to η_x, η_z, ξ_y while

$$\exp(-b\Pi - b'\Pi^*) = \exp[ij(b-b')(z-ct)]\exp\left(-\kappa x^2/|\omega|^2\right) \tag{B.2}$$

with

$$\kappa = a(b+b') + im(b-b')(z+ct), \quad |\omega|^2 = a^2 + (z+ct)^2. \tag{B.3}$$

Substituting (B.2) into (B.1) gives

$$U = \iint_0^\infty \mathrm{d}b\,\mathrm{d}b' F(b)F^*(b')\int_{-\infty}^\infty \mathrm{d}z \exp[ij(b-b')(z-ct)] \times \int_{-\infty}^\infty \mathrm{d}xG^\circ(x) \tag{B.4}$$

where

$$G^\circ(x) = |\omega|^{-1}P(b,b')\exp\left(-\kappa x^2/|\omega|^2\right). \tag{B.5}$$

And using the expressions (11) of η_x, η_z, ξ_y the same calculation as in (A.9) gives

$$P(b,b') = 2j^2bb' + m^2/2|\omega|^2 + (4bb' - \kappa|\omega|^{-2})x^2/|\omega|^2 + 2bb'm^2x^4/|\omega|^4. \tag{B.6}$$

Substituting (B.6) into (B.5) and using (A.11) gives

$$\int_{-\infty}^\infty \mathrm{d}xG^\circ(x) = 2bb'(\pi/\lambda)^{1/2}\left(j^2 + \kappa^{-1} + 3m^2\kappa^{-2}/4\right) + (m^2-1)bb'/|\omega|^2. \tag{B.7}$$

Taking into account (B.7) we write (B.4) $U = U_1 + U_2$ with $a = z + ct$

$$U_1 = 2\pi^{1/2}\iint_0^\infty \mathrm{d}b\,\mathrm{d}b'bb'F(b)F^*(b')J_1(b,b') \tag{B.8}$$

$$J_1(b,b') = \exp[-2ij(b-b')ct]\int_{-\infty}^\infty \mathrm{d}\alpha \exp[ij(b-b')\alpha] \times \left(j^2 + \kappa^{-1} + 3m^2\kappa^{-2}/4\right) \tag{B.9}$$

and

$$U_2 = 2\pi^{1/2}(m^2-1)\iint_0^\infty \mathrm{d}b\,\mathrm{d}b'bb'F(b)F^*(b')J_2(b,b') \tag{B.10}$$

$$J_2(b,b') = \exp[-2ij(b-b')ct]\int_{-\infty}^\infty \mathrm{d}\alpha\left(a^2 + m^2\alpha^2\right)^{-1} \times \exp[ij(b-b')\alpha]. \tag{B.11}$$

Proceeding as for (A.16) and using (A.18), (A.19) we get from (B.8), (B.9)

$$U_1 = 4\pi^{3/2}(2a)^{-1/2}\int_0^\infty \mathrm{d}b\, b^2|F(b)|^2 \times \left[(2-m^2)b^{3/2} + m^2b^{1/2}/4a + p(b)\right] \tag{B.12}$$

$$p(b) = \left(2a^{-1/2}b^2\right)\left(j^2 + m^2 - 2\right)\int_0^\infty \mathrm{d}y \times \exp(-y)[(y+2ab)^{-1/2}. \tag{B.13}$$

Now the integral in (B.11) is [21]

$$\int_{-\infty}^\infty \mathrm{d}\alpha\left(a^2 + m^2\alpha^2\right)^{-1}\exp[ij(b-b')\alpha] = (\pi/am)\exp(-aj|b-b'|/m). \tag{B.14}$$

Then, substituting (B.11) into (B.10) and taking into account (B.14) gives

$$U_2 = 2\pi^{3/2}\left[(m^2-1)/am\right]\iint_0^\infty \mathrm{d}b\,\mathrm{d}b'\,bb'F(b)F^*(b') \times \exp[-h|b-b'|) \tag{B.15}$$

where

$$h = aj/m - 2ijct, \quad \mathrm{Re}h > 0 \tag{B.16}$$

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